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Variational Inequality with Fuzzy Convex Cone

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Abstract. In this study, we discuss one type of variational inequality problem with a fuzzy convex cone $\tilde{X}$, denoted by VI($\tilde{X}, f$). Different classes of fuzzy convex cones which are considered in different context of the problems will be discussed. According to the existence theorem, an approach derived from the concepts of multiple objective mathematical programming problems for solving the VI($\tilde{X}, f$) is proposed. An algorithm is developed to find its fuzzy optimal solution set with complexity analysis.

Key words: Variational inequality, Fuzzy convex cone, Solution procedure, Complexity analysis

1. Introduction

From the literature, we know that variational inequality can be applied to many problems, such as transportation planning, economic analysis, and so on [1–3]. Due to the vagueness involved in real world problems, variational inequality in a fuzzy environment becomes an important problem both in theory and in practice. Normally, a variational inequality problem is to find $x^* \in X$ such that the vector $f(x^*)$ must not make an obtuse angle with all feasible vectors emanating from x^*[6, 7]. Thus, by denoting VI(X, f), X is referred to the domain of the problem and f can be regarded as all the decision factors that affect the result. However, in addition to the uncertainty of the environment for a variational inequality problem which leads to certain degrees of fuzziness in decision factors, and results in fuzzy $\tilde{f}$, there is uncertainty of the resources that induces certain degrees of fuzziness in the domain, and results in fuzzy $\tilde{X}$. The problem VI($X, \tilde{f}$) has been discussed in [9].

Therefore, in this paper we focus on the study of the variational inequality problem with fuzzy cone $\tilde{X}$, denoted by VI($\tilde{X}, f$). Since the domain $\tilde{X}$ is fuzzy, the solutions of VI($\tilde{X}, f$) is fuzzy too. While $\tilde{X}$ is considered as a fuzzy convex cone, the concept of fuzzy set [4, 10] will be adopted to convert a VI($\tilde{X}, f$) into a crisp form. Then the developed solution method for crisp VI [8] is used for solution.

In Section 2, we introduce basic concepts of fuzzy set. In Section 3, the properties of the fuzzy convex cone will first be discussed. In Section 4, based on the defined VI($\tilde{X}, f$), we shall present an equivalent relationship between VI($\tilde{X}, f$) and generalized complementarity problem GCP($\tilde{X}, f$). Then, through a transformation, a multiple objective programming model is derived to facilitate the solution process. In Section 5, the existence theorem and solution procedure of VI($\tilde{X}$, f) are proposed. In Section 6, an algorithm for the general case is developed, with illustrative examples. Finally, in Section 7, the observations are summarised and conclusions are drawn.

2. Basic concepts of fuzzy set

The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those certainly belong to the set) and nonmembers (those certainly not do so). But a fuzzy set is different. Let us look at the following definition:

DEFINITION 2.1. [4] If X is a collection of objects denoted generically by x then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in X\}$$

where $\mu_{\tilde{a}} : x \longrightarrow [0, 1]$ is called the membership function or grade of membership of x in $\tilde{A}$.

Therefore, in a fuzzy set, a membership function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Large values denote higher degrees of set membership [4].

Furthermore, two operations of fuzzy sets are introduced as follows:

DEFINITION 2.2. [4] The membership function $\mu_{\tilde{C}}(x)$ of the intersection of $\tilde{C} = \tilde{A} \cap \tilde{B}$ is pointwise defined by $\mu_{\tilde{C}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$, $x \in X$.

DEFINITION 2.3. [4] The membership function $\mu_{\tilde{D}}(x)$ of the union of $\tilde{D} = \tilde{A} \cup \tilde{B}$ is pointwise defined by $\mu_{\tilde{D}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$, $x \in X$.

3. The properties of fuzzy convex cone $\tilde{X}$

After introduce the basic concepts of fuzzy set and its operations, we turn to investigate the properties of fuzzy convex cone $\tilde{X}$.

We discover that from the causes of a fuzzy $\tilde{X}$, we can identify the following four cases:

(1) $\tilde{X} = \{\tilde{D}x \geqslant 0\} = \{(x, \mu_{\tilde{X}}(x))|\tilde{D}x \geqslant 0\}$. That is, the fuzziness of $\tilde{X}$ is caused by the uncertain domain $\tilde{D}$.

(2) $\tilde{X} = \{Dx \gtrsim 0\} = \{(x, \mu_{\tilde{X}}(x))|Dx \gtrsim 0\}$: the fuzziness of $\tilde{X}$ is caused by the uncertain relation $\gtrsim$.

(3) $\tilde{X} = \{Dx \geqslant \tilde{0}\} = \{(x, \mu_{\tilde{X}}(x))|Dx \geqslant \tilde{0}\}$: the fuzziness of $\tilde{X}$ is caused by the phenomenon that the vector x "almost" makes an acute angle with every row vector of D represented by right hand side $\tilde{0}$.

(4) $\tilde{X} = \{\tilde{D}x \underset{\sim}{\geqslant} \tilde{0}\} = \{(x, \mu_{\tilde{X}}(x))|\tilde{D}x \underset{\sim}{\geqslant} \tilde{0}\}$ or $\tilde{X} = \{\tilde{D}x \underset{\sim}{\geqslant} 0\} = \{(x, \mu_{\tilde{X}}(x))| \tilde{D}x \underset{\sim}{\geqslant} 0\}$ or $\tilde{\tilde{X}} = \{\tilde{D}x \geqslant \tilde{0}\} = \{(x, \mu_{\tilde{X}}(x))|\tilde{D}x \geqslant \tilde{0}\}$. The fuzziness of $\tilde{X}$ is caused by all uncertanties of the domain $\tilde{D}$, relation $\underset{\sim}{\geqslant}$ and the right hand side $\tilde{0}$.

Now, the properties of these four definitions and their relations will be shown by the following theorems.

THEOREM 3.1. *The set of $\tilde{X} = \{\tilde{D}x \geqslant 0\}$ is a convex cone where $\tilde{D} = [\tilde{D}_{ij}] = [\tilde{d}_i]$ is a fuzzy $m \times n$ matrix with $m \neq 1$.*

Proof. $\tilde{X} = \{\tilde{D}x \geqslant 0\} = \{(x, \mu_{\tilde{X}}(x)) : x \in \mathbf{R}^n$ such that $\mu_{\tilde{X}}(x) = \text{Min}_i\{\text{Min}_{d_i x \geqslant 0}\text{Sup}_{d_i x}\{\text{Min}_j\, \mu_{\tilde{D}}(D_{ij})\}\} \neq 0\}$. Let $\tilde{x}_1 = (x_1, \mu_{\tilde{X}}(x_1)) \in \tilde{X}$ and $\tilde{x}_2 = (x_2, \mu_{\tilde{X}}(x_2)) \in \tilde{X}$, consider $\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2$, $\lambda \in [0, 1]$.

$$\mu_{\tilde{X}}(\lambda x_1 + (1-\lambda)x_2) = \underset{i}{\text{Min}} \left\{ \underset{d_i(\lambda x_1 + (1-\lambda)x_2) \geqslant 0}{\text{Min}} \ \underset{d_i(\lambda x_1 + (1-\lambda)x_2)}{\text{Sup}} \{\mu_{\tilde{D}}(d_i)\} \right\}$$

$$= \underset{i}{\text{Min}} \left\{ \underset{\substack{d_i x_1 \geqslant 0 \\ d_i x_2 \geqslant 0}}{\text{Min}} \ \underset{\substack{d_i x_1 \\ d_i x_2}}{\text{Sup}}\{\mu_{\tilde{D}}(d_i)\} \right\} \geqslant \text{Min}\{\mu_{\tilde{X}}(x_1), \mu_{\tilde{X}}(x_2)\}$$

Let $\mu(d_i^* x_1) = \text{Min}_{d_i x_1 \geqslant 0}\{\mu(d_i x_1)\}$ and $\mu(d_i^{**} x_2) = \text{Min}_{d_i x_2 \geqslant 0}\{\mu(d_i x_2)\}$ for $i = 1, 2, \ldots, m$, then consider the following four conditions:

(1) Suppose $d_k^* x_1 \geqslant 0, d_k^{**} x_2 \geqslant 0, d_k^* x_2 \leqslant 0, d_k^{**} x_1 \leqslant 0$ for some $k \in \{1, 2, \ldots, m\}$, and $d_i^* x_1 \geqslant 0$, $d_i^* x_2 \geqslant 0$, $d_i^{**} x_1 \geqslant 0$, $d_i^{**} x_2 \geqslant 0$, $i \neq k$. Let $G_1 = \{x|d_i^* x \geqslant 0, i \neq k, d_k^* x \geqslant 0\}$, $G_2 = \{x|d_i^* x \geqslant 0, i \neq k, d_k^{**} x \geqslant 0\}$, $G_3 = \{x|d_i^{**} x \geqslant 0, i \neq k, d_k^* x \geqslant 0\}$ and $G_4 = \{x|d_i^{**} x \geqslant 0, i \neq k, d_k^{**} x \geqslant 0\}$. Since $d_k^* x_1 \geqslant 0$ and $d_k^{**} x_1 \leqslant 0$, hence $G_1 \supset G_2$. Since $d_k^{**} x_2 \geqslant 0$ and $d_k^* x_2 \leqslant 0$, hence $G_3 \subset G_4$. Then, $x_1 \in G_1$, $x_1 \in G_3$, $x_1 \in G_4$ and $d_k^{**} x_1 \geqslant 0$, a contradiction. Therefore, this condition does not exist.

(2) Suppose $d_k^* x_1 \geqslant 0, d_k^{**} x_2 \geqslant 0, d_k^* x_2 \geqslant 0, d_k^{**} x_1 \geqslant 0$ for some $k \in \{1, 2, \ldots, m\}$ and $d_i^* = d_i^{**}$, for all $i \neq k$. Since $\mu(d_i^* x_1) = \text{Min}_{d_i x_1 \geqslant 0}\{\mu(d_i x_1)\}$ for $i = 1, 2, \ldots, m$, $\mu(d_k^*) \geqslant \mu(d_k^{**})$. Hence $\mu(d_k^*) = \mu(d_k^{**})$.

(3) Suppose $d_k^* x_1 \geqslant 0, d_k^{**} x_2 \geqslant 0, d_k^* x_2 \geqslant 0, d_k^{**} x_1 \leqslant 0$ for some $k \in \{1, 2, \ldots, m\}$ and $d_i^* = d_i^{**}$, for all $i \neq k$. Then $\mu(d_k^*) > \mu(d_k^{**})$ and

$$\underset{d_i(\lambda x_1 + (1-\lambda)x_2) \geqslant 0}{\text{Min}}\{\mu(d_i(\lambda x_1 + (1-\lambda)x_2))\} = \underset{\substack{d_i x_1 \geqslant 0 \\ d_i x_2 \geqslant 0}}{\text{Min}}\{\mu(d_i)\} = \begin{cases} \mu(d_i^*) i \neq k \\ \mu(d_k^*) \end{cases}.$$

(4) Suppose $d_{k_l}^* x_1 \geqslant 0$, $d_{k_l}^{**} x_2 \geqslant 0$, $d_{k_l}^* x_2 \geqslant 0$, $d_{k_l}^{**} x_1 \geqslant 0$, $d_{j_l}^* x_1 \geqslant 0$, $d_{j_l}^{**} x_1 \leqslant 0$, $d_{j_l}^* x_2 \geqslant 0$, $d_{j_l}^{**} x_2 \geqslant 0$ for some $k, j \in \{1, 2, \ldots, m\}$, $l = 1, 2, \ldots, l'$, $l' \in$

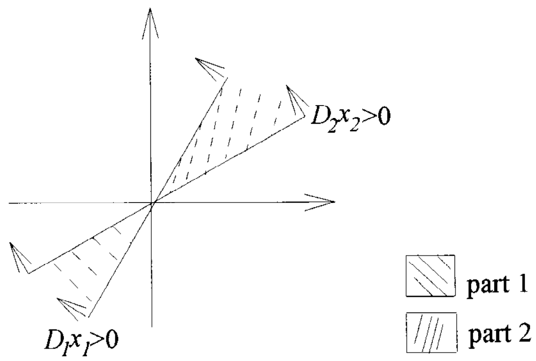

Figure 1.

$\{1, 2, \ldots, m\}$ and $d_i^* = d_i^{**}$, $i \neq k$, $i \neq j$. Then

$$\underset{d_i(\lambda x_1+(1-\lambda)x_2)\geqslant 0}{\mathrm{Min}} \{\mu(d_i(\lambda x_1 + (1-\lambda)x_2))\} = \underset{\substack{d_i x_1 \geqslant 0 \\ d_i x_2 \geqslant 0}}{\mathrm{Min}} \{\mu(d_i)\} = \mu(d_i^*).$$

Therefore, $\mu_{\tilde{X}}(\lambda x_1 + (1-\lambda)x_2)$ does exist. Furthermore, $\tilde{0} \in \tilde{X}$ and $(\lambda x_1, \mu_{\tilde{X}}(\lambda x_1)) = (\lambda x_1, \mu_{\tilde{X}}(x_1)) \in \tilde{X}$ for $\lambda > 0$, hence the set of $\tilde{X}$ is a convex cone.

However, when $m = 1$, $\tilde{D}$ is a fuzzy vector not fuzzy matrix. Let $\mu_{\tilde{X}}(x_1) = \mathrm{Min}_{Dx_1 \geqslant 0}\{\mu(DX_1)\} = \mu(D_1x_1)$ and $\mu_{\tilde{X}}(x_2) = \mathrm{Min}_{Dx_2 \geqslant 0}\{\mu(DX_2)\} = \mu(D_2x_2)$.

Suppose $D_1x_1 \geqslant 0$, $D_1x_2 \leqslant 0$, $D_2x_2 \geqslant 0$, $D_2x_1 \leqslant 0$. Then from Figure 1, we notice that x_1 is in part I and x_2 is in part II. Therefore, we can not find a vector D^*, $\mu_{\tilde{D}}(D^*) > 0$ such that $D^*x_1 \geqslant 0$ and $D^*x_2 \geqslant 0$ for all x_1, x_2, that is, $D^*(\lambda x_1 + (1-\lambda)x_2) \geqslant 0$, $\lambda \in [0, 1]$. Therefore, when $m = 1$, $\tilde{X}$ is not convex. □

COROLLARY 3.1. *If $\tilde{X} = \{\tilde{D}x \geqslant 0\}$ where $\tilde{D} = [\tilde{D}_{ij}]$ is a fuzzy $m \times n$ matrix and $m \neq 1$, then there exists a $m \times n$ matrix $A = [A_{ij}] = [a_i]$ such that $\tilde{X} = \{A\tilde{x} \geqslant 0\}$ where the row vector a_i, $i = 1, 2, \ldots, m$, is the optimal solution of the following system.*

$$\begin{aligned} \text{Max} \quad & \sum_{j=1, j\neq i}^{m} \left[\left(\sum_{k=1}^{n} (A_{ik} \times D_{jk}) \right) - \left(\sum_{k=1}^{n} D_{jk}^2 \right) \left(\sum_{k=1}^{n} A_{jk}^2 \right) \right] \\ \text{s.t.} \quad & \mu_{\tilde{D}_{ik}}(A_{ik}) > 0, \quad k = 1, 2, \ldots, n \end{aligned} \tag{1}$$

Proof. At first, we consider the set $\tilde{X}'' = \{A\tilde{x} \geqslant 0\}$. Let $\tilde{x}_1 = (x_1, \mu_{\tilde{X}}(x_1)) \in \tilde{X}''$ and $\tilde{x}_2 = (x_2, \mu_{\tilde{X}}(x_2)) \in \tilde{X}''$. That is, $A(x_1, \mu_{\tilde{X}}(x_1)) \geqslant (0, \mu_1)$ and $A(x_2, \mu_{\tilde{X}}(x_2)) \geqslant (0, \mu_2)$. Then consider $\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2$, $0 \leqslant \lambda \leqslant 1$. $A(\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2) = \lambda A(x_1, \mu_{\tilde{X}}(x_1)) + (1-\lambda)A(x_2, \mu_{\tilde{X}}(x_2)) \geqslant (0, \min\{\mu_1, \mu_2\})$. Since $\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2 \in \tilde{X}''$, hence $\tilde{X}''$ is a convex set. $\tilde{0} = (0, \mu_{\tilde{X}}(0)) \in \tilde{X}''$ and $A(\lambda\tilde{x}_1) = \lambda(A\tilde{x}_1) \geqslant (0, \mu_1)$ for $\lambda > 0$, $\lambda\tilde{x}_1 \in \tilde{X}$, therefore $\tilde{X}''$ is a convex cone.

Since the set of all the elements of $\tilde{X} = \{\tilde{D}x \geqslant 0\}$ is a convex cone where $\tilde{D} = [\tilde{D}_{ij}] = [\tilde{d}_i]$ is a fuzzy $m \times n$ matrix and $m \neq 1$, there exists a crisp $m \times n$ matrix A such that $\mu_{\tilde{D}}(A) \neq 0$ and $Ax \geqslant 0$ for all x satisfying $\mu_{\tilde{X}}(x) \neq 0$. Let $A = [a_i^*]_{i=1}^m = [A_{ij}^*]$ and $\mu_{\tilde{D}}(D_{ij}) = 1$. Then the row vector a_i^* is an optimal solution of $\{\mathrm{Max}(a_i \cdot d_j/\|a_i\|\,\|d_j\|),\ j \neq i,\ j = 1, 2, \ldots, m$ s.t. $\mu_{\tilde{D}}(a_i) \neq 0\}$, that is, a_i^* is an optimal solution of

$$\begin{aligned}
&\{\mathrm{Max}\, a_i \cdot d_j - \|a_i\|d_j\|,\ j = 1, 2, \ldots, m \text{ s.t. } \mu_{\tilde{D}}(a_i) > 0\}\\
&\quad = \left\{\mathrm{Max}\left[\left(\sum_{k=1}^{n}(A_{ik} \cdot D_{jk})\right) - \left(\sum_{k=1}^{n} D_{jk}^2\right)\left(\sum_{k=1}^{n} A_{jk}^2\right)\right],\right.\\
&\qquad\qquad j = 1, \ldots, m,\ j \neq i\\
&\qquad \left.\text{s.t. } \mu_{\tilde{D}}(A_{ik}) > 0, k = 1, \ldots, n\right\}\\
&\quad = \left\{\mathrm{Max} \sum_{j=1, j\neq i}^{m} \left[\left(\sum_{k=1}^{n}(A_{ik} \cdot D_{jk})\right) - \left(\sum_{k=1}^{n} D_{jk}^2\right)\left(\sum_{k=1}^{n} A_{jk}^2\right)\right],\right.\\
&\qquad \left.\text{s.t. } \mu_{\tilde{D}}(A_{ik}) > 0, \quad k = 1, \ldots, n\right\}
\end{aligned}$$

□

REMARK.

$$\begin{aligned}
\tilde{X} &= \{\tilde{D}x \geqslant 0\} = \{(x, \mu_{\tilde{D}}(x)) : (D, \mu_{\tilde{D}}(x))x \geqslant (0, \mu), \quad 0 \leqslant \mu \leqslant 1\}\\
&= \left\{(x, \mu_{\tilde{X}}(x)) : Ax \geqslant 0, \mu_{\tilde{X}}(x) = \mathop{\mathrm{Min}}_{i}\left\{\mathop{\mathrm{Min}}_{d_i x \geqslant 0}\ \mathop{\mathrm{Sup}}_{d_i x}\left\{\mathop{\mathrm{Min}}_{j}\ \mu_{\tilde{D}}(D_{ij})\right\}\right\}\right\}.
\end{aligned}$$

THEOREM 3.2.

$$\tilde{X} = \{Dx \underset{\sim}{\geqslant} 0\} = \{Dx \geqslant \tilde{0}\} = \left\{(x, \mu_{\tilde{X}}(x)) : \quad Dx \geqslant p, \quad p \leqslant 0,\right.$$

$$\left.\mu_{\tilde{X}}(x) = \mathop{\mathrm{Min}}_{i}\{\mu_i(x)\},\ \text{where } \mu_i(x) = \begin{cases} 0 & \text{if } d_i x \leqslant p_i \\ 1 - d_i x/p_i & \text{if } 0 > d_i x > p_i \\ 1 & \text{if } d_i x \geqslant 0 \end{cases}\right\}$$

Proof. By the definition. □

COROLLARY 3.2. *Suppose $\tilde{X} = \{DX \underset{\sim}{\geqslant} 0\}$ or $\tilde{X} = \{Dx \geqslant \tilde{0}\}$, then (a) if there exists a point x_0 such that $Dx_0 = p$ where p is the tolerance for $Dx \underset{\sim}{\geqslant} 0$ or $Dx \geqslant \tilde{0}$, then $\tilde{X}$ is a convex cone with respect to the point x_0; otherwise (b) $\tilde{X}$ is an convex set but not a cone.*

Proof. From theorem 3.2,

$$\tilde{X} = \{Dx \underset{\sim}{\geqslant} 0\} = \{Dx \geqslant \tilde{0}\} = \Bigg\{(x, \mu_{\tilde{X}}(x)) : \quad Dx \geqslant p, \quad p \leqslant 0,$$

$$\mu_{\tilde{X}}(x) = \operatorname*{Min}_i\{\mu_i(x)\}, \text{ where } \mu_i(x) = \begin{cases} 0 & \text{if } d_i x \leqslant p_i \\ 1 - d_i x/p_i & \text{if } 0 > d_i x > p_i \\ 1 & \text{if } d_i x \geqslant 0 \end{cases}\Bigg\}.$$

Hence, the set of all the elements of $\tilde{X}$ is a convex set. But, if there exists a point x_0 such that $Dx_0 = p$, then $D(x - x_0) \geqslant 0$.Therefore the set of $\tilde{X}$ is a convex cone with respect to the point x_0. □

THEOREM 3.3. *Let* $\tilde{X} = \{\tilde{D}x \underset{\sim}{\geqslant} 0\} = \{\tilde{D}x \geqslant p, p \leqslant 0\}$ *where* $\tilde{D} = [\tilde{D}_{ij}]$ *is a fuzzy* $m \times n$ *matrix. If there exists a point* x_0 *such that* $\tilde{D}x_0 = (p, \mu), 0 \leqslant \mu \leqslant 1$ *then the set of all the elements of* $\tilde{X}$ *is a convex cone with respect to* x_0.

Proof. Since $\tilde{D}x \geqslant p$, $p \leqslant 0$ and $\tilde{D}x_0 = (p, \mu)$, $0 \leqslant \mu \leqslant 1$, $\tilde{X} = \{\tilde{D}(x - x_0) \geqslant 0\}$ and the set of all the elements of $\tilde{X}$ is a convex cone with respect to x_0, then, from Corollary 3.1, $\tilde{X} = \{A(\tilde{x} - x_0) \geqslant 0\}$.

REMARK.

$$\begin{aligned} \tilde{X} &= \{\tilde{D}x \underset{\sim}{\geqslant} \tilde{0}\} = \{\tilde{D}x \underset{\sim}{\geqslant} 0\} = \{\tilde{D}x \geqslant \tilde{0}\} \text{ (since } \{Dx \underset{\sim}{\geqslant} 0\} = \{Dx \geqslant \tilde{0}\}) \\ &= \{\tilde{D}x \geqslant p, p \leqslant 0\} \\ &= \{(x, \mu_{\tilde{D}}(x)) : (D, \mu_{\tilde{D}}(x))x \geqslant (p, \mu), \quad 0 \leqslant \mu \leqslant 1\} \\ &= \Bigg\{(x, \mu_{\tilde{X}}(x)) : \qquad A(x - x_0) \geqslant 0, \end{aligned}$$

$$\mu_{\tilde{X}}(x) = \operatorname*{Min}_i \left\{ \operatorname*{Min}_{d_i x \geqslant p_i} \operatorname*{Sup}_{d_i x} \operatorname{Min} \left\{ \operatorname*{Min}_j \mu_{\tilde{D}}(D_{ij}),\ \mu_i(x) \right\} \right\}$$

$$\text{where } \mu_i(x) = \begin{cases} 0 & \text{if } d_i x \leqslant p_i \\ 1 - d_i x/p_i & \text{if } 0 > d_i x > p_i \\ 1 & \text{if } d_i x \geqslant 0 \end{cases}\Bigg\}.$$

Since from corollary 3.2, classes (2) and (3) have similar properties, therefore based on the developed solution method for crisp VI under a convex cone [4], in the following we consider the solutions of VI($\tilde{X}$, f) with convex cone $\tilde{X}$ in three existing cases: (1) $\tilde{X}_1 = \{\tilde{D}x \geqslant 0\}$; (2) $\tilde{X}_2 = \{Dx \underset{\sim}{\geqslant} 0\}$ or $\tilde{X}_2 = \{Dx \geqslant \tilde{0}\}$ in Corollary 3.2(a); and (3) $\tilde{X}_3 = \{\tilde{D}x \underset{\sim}{\geqslant} \tilde{0}\}$ in Theorem 3.3. (1) is a convex cone with respect to the origin, but (2) and (3) are convex cone with respect to some point that is not the origin. Therefore, we can define a general form to represent these three conditions as: $\tilde{X} = \{D'\tilde{x} \geqslant b\} = \{(x, \mu_{\tilde{X}}(x)) | D'\tilde{x} \geqslant b\ \forall \tilde{x} \in \tilde{X}, D'x_0 = b$

for $\tilde{x}_0 = (x_0, 1) \in \tilde{X}\}$. Hence, if $\tilde{X} = \tilde{X}_1$ then $D' = A$, $b = 0$ and $x_0 = 0$ where A is defined in Corollary 3.1; if $\tilde{X} = \tilde{X}_2$ then $D' = D$, $b = p$ and $x_0 \neq 0$; if $\tilde{X} = \tilde{X}_3$ then $D' = A$, $b = p$ and $x_0 \neq 0$.

4. Problem formulations

A variational inequality problems with fuzzy $\tilde{X}$ is to find $\tilde{x}^* \in \tilde{X}$ such that

$$\textbf{(VI(}\tilde{X}\textbf{,f))}\ \langle f(\tilde{x}^*), \tilde{x} - \tilde{x}^*\rangle \geqslant (0, \mu),\ 0 \leqslant \mu \leqslant 1,\ \text{for all } \tilde{x} \in \tilde{X} \qquad (2)$$

That is, in $\tilde{X}$, a vector x^* has the degree of membership $\mu_{\tilde{X}}(x^*)$, which satisfies that $\langle f(x^*), x - x^*\rangle \geqslant 0$, for all $x \in X$ to some degree μ, $0 \leqslant \mu \leqslant 1$. Hence, to some degree $\mu_{\tilde{X}}(x^*)$, x^* is an optimal solution and denoted by $(x^*, \mu_{\tilde{X}}(x^*))$. Thus, the optimal solution set is a fuzzy set.

Consider a generalized complementarity problem (GCP) with fuzzy domain as follows:

$$\begin{aligned}
&(\text{GCP}(\tilde{X}, f))\\
&\quad \text{For } f((x^*, \mu_{\tilde{X}}(x^*))) \in \tilde{X}^{\perp},\\
&\quad \langle f((x^*, \mu_{\tilde{X}}(x^*))), (x^*, \mu_{\tilde{X}}(x^*))\rangle = (0, \mu), 0 \leqslant \mu \leqslant 1, \qquad (3)\\
&\quad \text{where } \tilde{X}^{\perp} = \{(z, \mu_{\tilde{z}}(z)) : \langle (z, \mu_{\tilde{z}}(z)), (x, \mu_{\tilde{X}}(x))\rangle \geqslant (0, \mu) \text{ for all}\\
&\quad (x, \mu_{\tilde{X}}(x)) \in \tilde{X}\}
\end{aligned}$$

we have the following results.

THEOREM 4.1. *VI($\tilde{X}$, f)=GCP($\tilde{X}$, f), when the set of all the elements of $\tilde{X}$ is a convex cone.*

Proof. First, we show that VI$(\tilde{X}, f) \subset$ GCP$(\tilde{X}, f)$. Suppose that $(x^*, \mu_{\tilde{X}}(x^*))$ is an optimal solution of VI$(\tilde{X}, f)$. Then, when $x = 0$, we have $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x^*, \mu_{\tilde{X}}(x^*))\rangle \leqslant (0, \mu), 0 \leqslant \mu \leqslant 1$. Let $(x, \mu_{\tilde{X}}(x)) = \lambda(x^*, \mu_{\tilde{X}}(x^*))$ with $\lambda > 1$, we see that $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x^*, \mu_{\tilde{X}}(x^*))\rangle \geqslant (0, \mu)$. Therefore, $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x^*, \mu_{\tilde{X}}(x^*))\rangle = (0, \mu)$. Assume $f((x^*, \mu_{\tilde{X}}(x^*))) \notin \tilde{X}^{\perp}$, then there exist $(x', \mu_{\tilde{X}}(x')) \in \tilde{X}$ such that $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x', \mu_{\tilde{X}}(x'))\rangle < (0, \mu)$. But according to (2), $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x'\mu_{\tilde{X}}(x'))\rangle \geqslant \langle f((x^*, \mu_{\tilde{X}}(x^*))), x^*\rangle = (0, \mu)$. There lies a contradiction. Hence, $(x^*, \mu_{\tilde{X}}(x^*))$ is also an optimal solution of GCP$(\tilde{X}, f)$. Conversely, suppose $(x^*\mu_{\tilde{X}}(x^*))$ is also an optimal solution of GCP$(\tilde{X}, f)$, then $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x, \mu_{\tilde{X}}(x))\rangle \geqslant (0, \mu)$ for all $(x, \mu_{\tilde{X}}(x)) \in \tilde{X}$ and $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x, \mu_{\tilde{X}}(x)) - (x^*, \mu_{\tilde{X}}(x^*))\rangle \geqslant (0, \mu)$. Hence, GCP$(\tilde{X}, f) \subset$ VI$(\tilde{X}, f)$. Therefore, VI$(\tilde{X}, f) =$ GCP$(\tilde{X}, f)$ when $\tilde{X}$ is a convex cone. □

Theorem 4.1 states an equivalence between a fuzzy generalized complementarity problem (GCP) and a fuzzy variational inequality problem (VI) over a fuzzy convex cone. Since the concepts of multiple objective programming (MOP) have been employed to solve a generalized complementarity problem [8], we shall show that the analogous approach can be extented to fuzzy cases by formulating a GCP($\tilde{X}, f$) in the form of fuzzy MOP.

$$\begin{aligned}
&\text{Fuzzy-MOP}(\tilde{X}, f)\\
&\qquad \text{Minimize } [\tilde{y}_1\tilde{x}_1, \tilde{y}_2\tilde{x}_2, \ldots, \tilde{y}_n\tilde{x}_n]^{\mathbf{T}}\\
&\qquad \text{subject to } (x, \mu_{\tilde{X}}(x)) \in \tilde{X}\\
&\qquad\quad f((x^*, \mu_{\tilde{X}}(x^*))) \in \tilde{X}^{\perp}\\
&\qquad\quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)^{\mathbf{T}} = f((x, \mu_{\tilde{X}}(x)))
\end{aligned} \tag{4}$$

THEOREM 4.2. *Let $\tilde{X}_{eff}$ denote the set of all efficient solutions of a* Fuzzy-MOP*($\tilde{X}, f$) defined in (4). Then the point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathbf{T}}$ with some $\mu_{\tilde{X}}(x^*)$ belonging to $\tilde{X}_{eff}$ is a solution of the* Fuzzy-MOP*($\tilde{X}, f$) such that $\tilde{y}_1^*\tilde{x}_1^* + \tilde{y}_2^*\tilde{x}_2^* + \cdots + \tilde{y}_n^*\tilde{x}_n^* = (0, \mu), 0 \leqslant \mu \leqslant 1$, if and only if $(x^*, \mu_{\tilde{X}}(x^*))$ is a solution of GCP($\tilde{X}, f$).*

Proof. Suppose that (i) $(x^*, \mu_{\tilde{X}}(x^*)) \in \tilde{X}_{eff}$ and (ii) $\tilde{y}_1^*\tilde{x}_1^* + \tilde{y}_2^*\tilde{x}_2^* + \cdots + \tilde{y}_n^*\tilde{x}_n^* = (0, \mu), 0 \leqslant \mu \leqslant 1$. According to (i), $(x^*, \mu_{\tilde{X}}(x^*)) \in \tilde{X}$ and $f((x^*, \mu_{\tilde{X}}(x^*))) \in \tilde{X}^{\perp}$. By (ii), x^* with $\mu_{\tilde{X}}(x^*)$ such that $\langle f((x^*, \mu_{\tilde{X}}(x^*))), (x^*, \mu_{\tilde{X}}(x^*))\rangle = (0, \mu)$, $0 \leqslant \mu \leqslant 1$. Then $(x^*, \mu_{\tilde{X}}(x^*))$ is an optimal solution of GCP($\tilde{X}, f$). Conversely, suppose that $(x^*, \mu_{\tilde{X}}(x^*))$ is an optimal solution of GCP($\tilde{X}, f$), then, according to definition of GCP($\tilde{X}, f$), $(x^*, \mu_{\tilde{X}}(x^*)) \in \tilde{X}_{eff}$ and $\tilde{y}_1^*\tilde{x}_1^* + \tilde{y}_2^*\tilde{x}_2^* + \cdots + \tilde{y}_n^*\tilde{x}_n^* = (0, \mu), 0 \leqslant \mu \leqslant 1$. □

By the above theorem, we know the relation between GCP($\tilde{X}, f$) and fuzzy MOP. Now, we connect two types of $\tilde{X}$ defined in the previous section, into a fuzzy MOP model. Then, the model is transformed into a single objective programming model for ease of solutions.

THEOREM 4.3. *Consider $\tilde{X} = \{D'\tilde{x} \geqslant b\}$ with a point x_0 which satisfies $D'x_0 = b$. Then the point $(x^*, \mu_{\tilde{X}}(x^*)) \in \tilde{X}_{eff}$ is a solution of the* Fuzzy-MOP($\tilde{X}, f$) *such that $\tilde{y}_1^*\tilde{x}_1^* + \tilde{y}_2^*\tilde{x}_2^* + \cdots + \tilde{y}_n^*\tilde{x}_n^* = (0, \mu)$, $0 \leqslant \mu \leqslant 1$, if and only if $(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (x^* - x_0, \mu_{\tilde{X}}(x^*))$ is a solution of model (5) defined below such that $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu)$, $0 \leqslant \mu \leqslant 1$.*

$$\begin{aligned}
&(\text{GCP}'(\tilde{X}, f))\\
&\quad \text{Minimize } (\nu, \mu')^{\mathbf{T}} D'(s, \mu_{\tilde{X}}(s + x_0))\\
&\quad \text{subject to } D'(s, \mu_{\tilde{X}}(s + x_0)) \geqslant (0, \mu'')\\
&\qquad\qquad f((s + x_0, \mu_{\tilde{X}}(s + x_0))) = D'^{\mathbf{T}}(\nu, \mu')\\
&\qquad\qquad (\nu, \mu') \geqslant (\mathbf{0}, \mu')\\
&\qquad\qquad 0 \leqslant \mu' \leqslant 1,\ 0 \leqslant \mu'' \leqslant 1.
\end{aligned} \tag{5}$$

Proof. Since $\tilde{X} = \{D'\tilde{x} \geqslant b\}$ satisfies that there exists a point x_0 such that $D'x_0 = b$, $\tilde{X}$ is a convex cone with respect to the point x_0. If we translate $\tilde{X}$ to the origin $\tilde{s} = (s, \mu_{\tilde{X}'}(s)) = (x - x_0, \mu_{\tilde{X}}(x))$, then we have a convex cone $\tilde{X}' = \{D'\tilde{s} \geqslant 0\}$. Then, $(x^*, \mu_{\tilde{X}}(x^*)) = (s^* + x_0, \mu_{\tilde{X}'}(s^*))$, where $(x^*, \mu_{\tilde{X}}(x^*))$ is the optimal solution of VI($\tilde{X}, f$), and $(s^* + x_0, \mu_{\tilde{X}'}(s^*))$ is the optimal solution of VI($\tilde{X}', f'$) with $f'(s) = f(s + x_0)$. That is, $\mu_{\tilde{X}'}(s^*) = \mu_{\tilde{X}}(s^* + x_0)$. And then, the equivalent GCP($\tilde{X}', f'$)(3) can be transformed into Fuzzy-MOP($\tilde{X}', f'$). Now, $D'(s^*, \mu_{\tilde{X}}(s^*+x_0)) \geqslant 0$, $\langle f((s^*+x_0, \mu_{\tilde{X}}(s^*+x_0))), (s, \mu_{\tilde{X}}(s+x_0))\rangle \geqslant (0, \mu)$ for all $\tilde{s} \in \tilde{X}'$, hence there exists some $(\nu, \mu') \geqslant (\mathbf{0}, \mu')$ such that $f((s^*+x_0, \mu_{\tilde{X}}(s^*+x_0))) = D'^{\mathbf{T}}(\nu, \mu')$. Therefore, when $\tilde{X} = \{D'\tilde{x} \geqslant b\}$ with a point x_0 which satisfies $D'x_0 = b$, the Fuzzy-MOP($\tilde{X}, f$) can be transformed into GCP$'$($\tilde{X}, f$). □

COROLLARY 4.1. *Let $\tilde{X} = \{D'\tilde{x} \geqslant b\}$ with a point x_0 which satisfies $D'x_0 = b$. Then $(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (x^* - x_0, \mu_{\tilde{X}}(x^*))$ is a solution of model (5) such that $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu)$, $0 \leqslant \mu \leqslant 1$, if and only if $(x^*, \mu_{\tilde{X}}(x^*))$ is a solution of* GCP($\tilde{X}, f$).

REMARK. (1) If $D' = A$, $b = 0$ and $x_0 = 0$ where A is defined in Corollary 3.1, then GCP$'$($\tilde{X}, f$) is equivalent to GCP($\tilde{X}_1, f$) and VI($\tilde{X}_1, f$); (2) If $D' = D$, $b = p$ and $x_0 = 0$, then GCP$'$($\tilde{X}, f$) is equivalent to GCP($\tilde{X}_2, f$) and VI($\tilde{X}_2, f$); (3) If $D' = A$, $b = p$ and $x_0 \neq 0$ where A is defined as Corollary 3.1, then GCP$'$($\tilde{X}, f$) is equivalent to GCP($\tilde{X}_3, f$) and VI($\tilde{X}_3, f$).

5. Solutions

From the previous section, we know the relation between GCP($\tilde{X}, f$) and MOP under two types of fuzzy convex cone. Next, we apply the reformulated GCP$'$($\tilde{X}, f$) to solve GCP($\tilde{X}, f$) and VI($\tilde{X}, f$).

THEOREM 5.1. *Let $\tilde{X}\{D'\tilde{x} \geqslant b\}$ with a point x_0 satisfy $D'x_0 = b$, $(s^*, \mu_{\tilde{X}}(s^* + x_0))$ is an optimal solution of GCP$'$($\tilde{X}, f$) such that $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^*+x_0)) = (0, \mu)$, for some $\mu \in [0, 1]$. Then,*

(i) $(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (\mathbf{0}, \mu_{\tilde{X}}(x_0))$ if and only if $f((x_0, \mu_{\tilde{X}}(x_0))) = D'^{\mathbf{T}}(\nu, \mu')$, $(\nu, \mu') \geqslant (0, \mu')$; or,

(ii) $s^* \neq \mathbf{0}$, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = (\mathbf{0}, \mu_{\tilde{X}}(f(x^*)))$ if and only if $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \geqslant (0, \mu'')$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \neq (0, \mu'')$, $0 \leqslant \mu'' \leqslant 1$; or

(iii) $s^* \neq \mathbf{0}$, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) \neq (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$ if and only if (1) there exists a $k \in \{1, 2, \ldots, m\}$ such that $d'_k(s^*, \mu_{\tilde{X}}(s^*+x_0)) = (0, \mu_k)$, $d'_i(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_i)$, $\mu_k, \mu_i \in [0, 1]$, for all $i = 1, 2, \ldots, m$, $i \neq k$, where d'_i's are row vectors of D', and there exists a $l > 0$, $l \in \mathbf{R}$, such that $f((s^*+x_0, \mu_{\tilde{X}}(s^*+x_0))) = (l \cdot d'_k, \mu_{\tilde{F}}(f(x^*)))$; or (2) if $m > 2, n > 2$, there exist k_1 and k_2, $k_1 \neq k_2$ and $k_1, k_2 \in \{1, 2, \ldots, m\}$ such that $d'_{k_1}(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_1})$, $d'_{k_2}(s^*, \mu_{\tilde{X}}(s^* +$

$x_0)) = (0, \mu_{k_2})$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_i)$ *for all* $i = 1, 2, \ldots, m, i \neq k_1, k_2$, *and there exist some* $l_1 > 0$ *and* $l_2 > 0$, $l_1, l_2 \in \mathbf{R}$, *such that* $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = ((l_1 d_{k_1}' + l_2 d_{k_2}'), \mu_{\tilde{F}}(f(x^*)))$.

Proof. Suppose $(s^*, \mu_{\tilde{X}}(s^*+x_0))$ is an optimal solution of $\mathrm{GCP}'(X, \tilde{f})$ such that $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu)$, for some $\mu \in (0, 1]$. Let $\tilde{y} = D'^{\mathbf{T}}(\nu, \mu')$ for some μ', then $\tilde{y}^{\mathbf{T}}(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu)$. So the optimal solution can be found when $(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (\mathbf{0}, \mu_{\tilde{X}}(x^*))$ $\tilde{y} = (0, \mu_{\tilde{F}}(f(s^* + x_0)))$ or $s^* \neq \mathbf{0}$ and $\tilde{y} \neq (\mathbf{0}, \mu_{\tilde{F}}(f(s^* + x_0)))$ where $\tilde{y}$ is normal to $(s^*, \mu_{\tilde{X}}(s^* + x_0))$.
(i) $(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (\mathbf{0}, \mu_{\tilde{X}}(x_0))$ then $f((x_0, \mu_{\tilde{X}}(x_0))) = D'^{\mathbf{T}}(\nu, \mu')$ and $(\nu, \mu') \geqslant (\mathbf{0}, \mu')$. Hence, if D'^{-1} exists, then $f((x_0, \mu_{\tilde{X}}(x_0)))^{\mathbf{T}} D'^{-1} \geqslant (\mathbf{0}, \mu')^{\mathbf{T}}$.
(ii) $s^* \neq \mathbf{0}$ and $\tilde{y} = (0, \mu_{\tilde{F}}(f(x^*)))$, then $D'^{\mathbf{T}}(\nu, \mu') = (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$, $(\nu, \mu') = (\mathbf{0}, \mu')$. Since $(s^*, \mu_{\tilde{X}}(s^*+x_0)))$ is optimal to (5) and $s^* \neq \mathbf{0}$, $D'(s^*, \mu_{\tilde{X}}(s^*+x_0)) \geqslant (0, \mu'')$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \neq (0, \mu'')$.
(iii) $s^* \neq \mathbf{0}$, and $\tilde{y} \neq (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$ and $\tilde{y}$ is normal to $(s^*, \mu_{\tilde{X}}(s^* + x_0))$, then $(\nu, \mu') \geqslant (\mathbf{0}, \mu')$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \geqslant (0, \mu'')$ and $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu)$. Suppose $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_t)$ for all $i = 1, 2, \ldots, m$ and let $(\nu, \mu')^{\mathbf{T}} = ((\nu_1, \mu_1'), (\nu_2, \mu_2'), \ldots, (\nu_m, \mu_m'))$. $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = ((\nu_1 d_1' s^* + \nu_2 d_2' s^* + \cdots + \nu_m d_m' s^*, \mu) = (0, \mu)$ and $\nu \neq \mathbf{0}$, then $\nu_j < 0$ for some j; this is a contradiction. Since $x^* - x_0 \neq \mathbf{0}$ and $\tilde{X}$ is a convex cone, hence (1) there exists a $k \in \{1, 2, \ldots, m\}$ such that $d_k'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_k)$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_i)$, $\mu_k, \mu_i \in [0, 1]$ for all $i = 1, 2, \ldots, m, i \neq k$ or (2) if $m > 2, n > 2$, there exist k_1 and k_2, $k_1 \neq k_2$ and $k_1, k_2 \in \{1, 2, \ldots, m\}$ such that $d_{k_1}'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_1})$, $d_{k_2}'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_2})$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_i)$, for all $i = 1, 2, \ldots, m, i \neq k_1, k_2$. Therefore, from (1), $(\nu, \mu')^{\mathbf{T}}$ $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = ((\nu_1 d_1' s^* + \nu_2 d_2' s^* + \cdots \nu_k \cdot 0 + \cdots + \nu_m d_m' s^*), \mu)$ $= (0, \mu)$, $\nu, \mu)^{\mathbf{T}} = ((0, \ldots, 0, l, 0, \ldots, 0), \mu')$, $l > 0$ is the kth element of ν, and $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = (l \cdot d_k', \mu_{\tilde{F}}(f(x^*)))$.

Conversely,
(a) If $f((x_0, \mu_{\tilde{X}}(x_0))) = D'^{\mathbf{T}}(\nu, \mu')$ and $(\nu, \mu') \geqslant (0, \mu')$ for some μ', consider the point $s^* = 0$. $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu'')$, hence $(x_0, \mu_{\tilde{X}}(x_0))$ is an optimal solution, and $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu)$, for some $\mu \in [0, 1]$.
(b) If $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \geqslant (0, \mu'')$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \neq (0, \mu'')$ and $(s^*, \mu_{\tilde{X}}(s^* + x_0))$ is an optimal solution such that $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0))$ $= (0, \mu)$, $\mu \in [0, 1]$, then $(\nu, \mu') = (\mathbf{0}, \mu')$. Hence, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0)))$ $= D'^{\mathbf{T}}(\nu, \mu') = (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$.
(c) (1) Suppose there exists a $k \in \{1, 2, \ldots, m\}$ such that $d_k'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_k)$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_t)$, $\mu_k, \mu_i \in [0, 1]$ for all $i = 1, 2, \ldots, m, i \neq k$, where d_i''s are row vectors of D', and $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = (l \cdot d_k', \mu_{\tilde{F}}(f(x^*)))$, $l > 0, l \in \mathbf{R}$. Then $x^* \neq \mathbf{0}$ and $\tilde{y} \neq (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = ((d_1' s^*, \ldots, d_{k-1}' s^*, 0, d_{k+1}' s^*, \ldots, d_m'(s^*)^{\mathbf{T}}, \mu'') > 0$, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = D'^{\mathbf{T}}(\nu, \mu') = (l \cdot d_k', \mu_{\tilde{F}}(f(x^*)))$, hence $(\nu, \mu')^{\mathbf{T}} = ((0, \ldots, 0, l_1, 0, \ldots, 0, l_2, 0, \ldots, 0), \mu') > (\mathbf{0}, \mu')$, $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^*+x_0)) =$

$((\nu_1 d_1' s^* + \nu_2 d_2' s^* + \cdots + \nu_k \cdot 0 + \cdots + \nu_m d_m' s^*), \mu) = (0, \mu)$. Hence, $(s^*, \mu_{\tilde{X}}(s^* + x_0))$ is an optimal solution.

(2) If $m > 2, n > 2$, suppose there exist k_1 and k_2, $k_1 \neq k_2$ and $k_1, k_2 \in \{1, 2, \ldots, m\}$ such that $d_{k_1}'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_1})$, $d_{k_2}'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_2})$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_i)$ for all $i = 1, 2, \ldots, m$, $i \neq k_1, k_2$, and there exist $l_1, l_2 > 0$, $l_1, l_2 \in \mathbf{R}$, such that $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = ((l_1 d_{k_1}' + l_2 d_{k_2}'), \mu_{\tilde{F}}(f(x^*)))$. Then $s^* \neq \mathbf{0}$ and $\tilde{y} \neq (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = ((d_1 s^*, \ldots, d_{k_1-1} s^*, 0, d_{k_1+1} s^*), \ldots, d_{k_2-1} s^*, d_{k_2+1} s^*, \ldots, d_m s^*), \mu'') > (0, \mu'')$, $f((s^*, \mu_{\tilde{X}}(s^* + x_0))) = D'^{\mathbf{T}}(\nu, \mu') = (l_1 d_{k_1}' + l_2 d_{k_2}', \mu_{\tilde{F}}(f(x^*)))$, hence $(\nu, \mu')^{\mathbf{T}} = ((0, \ldots,$
$0, l, 0, \ldots, 0), \mu') > (\mathbf{0}, \mu')$, $(\nu, \mu')^{\mathbf{T}} D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (\nu_1 d_1' s^* + \cdots + \nu_{k_1} \cdot 0 + \cdots +$
$\nu_{k_2} \cdot 0 + \cdots + \nu_m d_m' s^*, \mu) = (0, \mu)$. Hence, $s^*, \mu_{\tilde{X}}(s^* + x_0))$ is an optimal solution. □

COROLLARY 5.1. *Let $\tilde{X} = \{D'x \geqslant b\}$ with a point x_0 satisfying $D'x_0 = b$. If $(x^*, \mu_{\tilde{X}}(x^*))$ solves GCP($\tilde{X}$, f) and $(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (x^* - x_0, \mu_{\tilde{X}}(x^*))$, then*
(i) $s^*, \mu_{\tilde{X}}(s^* + x_0)) = (\mathbf{0}, \mu_{\tilde{X}}(\mathbf{0}))$ if and only if $f((x_0, \mu_{\tilde{X}}(x_0))) = D'^{\mathbf{T}}(\nu, \mu')$, $(\nu, \mu') \geqslant (0, \mu')$; or,
(ii) $s^* \neq \mathbf{0}$, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$, if and only if $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \geqslant (0, \mu'')$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \neq (0, \mu'')$, $0 \leqslant \mu'' \leqslant 1$; or,
(iii) $s^* \neq \mathbf{0}$, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) \neq (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$, if and only if (1) there exists a $k \in \{1, 2, \ldots, m\}$ such that $d_k'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_k)$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_i)$, $\mu_k, \mu_i \in [0, 1]$, for all $i = 1, 2, \ldots, m$, $i \neq k$, where d_i''s are row vectors of D', and there exists a $l > 0$, $l \in \mathbf{R}$, such that $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = (l \cdot d_k', \mu_{\tilde{F}}(f(x^*)))$; or (2) if $m > 2, n > 2$, there exist k_1 and k_2, $k_1 \neq k_2$ and k_1, $k_2 \in \{1, 2, \ldots, m\}$ such that $d_{k_1}'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_1})$, $d_{k_2}'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_1})$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_i)$ for all $i = 1, 2, \ldots, m, i \neq k_1, k_2$, and there exist some $l_1 > 0$ and $l_2 > 0$, $l_1, l_2 \in \mathbf{R}$, such that $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = ((l_1 d_{k_1}' + l_2 d_{k_2}'), \mu_{\tilde{F}}(f(x^*)))$.

COROLLARY 5.2. *Consider $\tilde{X} = \{D'x \geqslant b\}$ with a point x_0 which satisfies $D'x_0 = b$. If $(x^*, \mu_{\tilde{X}}(x^*))$ solves VI($\tilde{X}$, f) (Equation (2)) and $(s^*.\mu_{\tilde{X}}(s^* + x_0)) = (x^* - x_0, \mu_{\tilde{X}}(x^*))$, then*
(i) $s^*, \mu_{\tilde{X}}(s^* + x_0)) = (\mathbf{0}, \mu_{\tilde{X}}(\mathbf{0}))$ if and only if $f((x_0, \mu_{\tilde{X}}(x_0))) = D'^{\mathbf{T}}(\nu, \mu')$, $(\nu, \mu') \geqslant (0, \mu')$; or,
(ii) $s^* \neq \mathbf{0}$, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$, if and only if $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \geqslant (0, \mu'')$, $D'(s^*, \mu_{\tilde{X}}(s^* + x_0)) \neq (0, \mu'')$, $0 \leqslant \mu'' \leqslant 1$; or,
(iii) $s^* \neq \mathbf{0}$, $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) \neq (\mathbf{0}, \mu_{\tilde{F}}(f(x^*)))$, if and only if (1) there exists a $k \in \{1, 2, \ldots, m\}$ such that $d_k'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_k)$, $d_i'(s^*, \mu_{\tilde{X}}(s^* + x_0)) > (0, \mu_i)$, $\mu_k, \mu_i \in [0, 1]$, for all $i = 1, 2, \ldots, m$, $i \neq k$, where d_i''s are row vectors of D', and there exists a $l > 0$, $l \in \mathbf{R}$, such that $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = (l \cdot d_k', \mu_{\tilde{F}}(f(x^*)))$; or (2) if $m > 2, n > 2$, there exist k_1 and k_2, $k_1 \neq k_2$ and k_1, $k_2 \in \{1, 2, \ldots, m\}$ such that $d_{k_1}'(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_{k_1})$, $d_{k_2}'(s^*, \mu_{\tilde{X}}(s^* +$

$x_0)) = (0, \mu_{k_1})$, $d'_i(s^*, \mu_{\tilde{X}}(s^* + x_0)) = (0, \mu_i)$ *for all* $i = 1, 2, \ldots, m, i \neq k_1, k_2$, *and there exist some* $l_1 > 0$ *and* $l_2 > 0$, $l_1, l_2 \in \mathbf{R}$, *such that* $f((s^* + x_0, \mu_{\tilde{X}}(s^* + x_0))) = ((l_1 d'_{k_1} + l_2 d'_{k_2}), \mu_{\tilde{F}}(f(x^*)))$.

REMARK. (1) Let $\tilde{X}_1 = \{\tilde{D}x \geqslant 0\}$. If $(x^*, \mu_{\tilde{X}}(x^*))$ solves VI$(\tilde{X}_1, f)$, then the solution satisfies Corollary 5.2 with $D' = A$, $b = 0$ and $x_0 = 0$ where A is defined as Corollary 3.1; (2) Let $\tilde{X}_2 = \{Dx \underset{\sim}{\geqslant} 0\}$ or $\tilde{X}_2 = \{Dx \geqslant \tilde{0}\}$ which satisfies Corollary 3.2(a). If $(x^*, \mu_{\tilde{X}}(x^*))$ solves VI$(\tilde{X}_2, f)$, then the solution satisfies Corollary 5.2 with $D' = D$, $b = p$ and $x_0 = 0$; (3) Let $\tilde{X}_3 = \{\tilde{D}x \underset{\sim}{\geqslant} \tilde{0}\}$ which satisfies Theorem 3.3. If $x^*, \mu_{\tilde{X}}(x^*))$ solves VI$(\tilde{X}_1, f)$, then the solution satisfies Corollary 5.2 with $D' = A$, $b = p$ and $x_0 = 0$ where A is defined as Corollary 3.1.

6. Algorithm and examples

Based on the theorems developed in the previous section, in this section, we proposed an algorithm for solving VI$(\tilde{X}, f)$.

Algorithm for VI$(\tilde{X}, f)$:

Step 0. Let $\tilde{X}^* = \emptyset$ be the optimal solution set of the VI$(\tilde{X}, f)$

Step 1.

Step 1.1. If $\tilde{X}_1 = \{\tilde{D}x \geqslant 0\}$, then solve the nonlinear system (1) by GINO[5] to obtain the optimal solution a_i^* for $i = 1, 2, \ldots, m$. Then let $D' = A = [a_i^*]$, $b = 0$ and $x_0 = 0$.

Step 1.2. If $\tilde{X}_2 = \{Dx \underset{\sim}{\geqslant} 0\}$ or $\tilde{X}_2 = \{Dx \geqslant \tilde{0}\}$ satisfies the Corollary 3.2(a), then let $D' = \tilde{D}$, $b = p$ and solve $Dx_0 = p$.

Step 1.3. If $\tilde{X}_3 = \{\tilde{D}x \underset{\sim}{\geqslant} \tilde{0}\}$ which satisfies Theorem 3.3, then solve the nonlinear system (1) by GINO[5] and get the optimal solution a_i^* for $i = 1, 2, \ldots, m$. Then let $D' = A = [a_i^*]$, $b = p$ and solve $Ax_0 = p$.

Step 2. If D' is a rectangular matrix, then solve the following system.

$$f((x_0, \mu_{\tilde{X}}(x_0))) = D'^{\mathbf{T}}(\nu, \mu'). \tag{6}$$

If $(\nu, \mu') \geqslant (\mathbf{0}, \mu')$ for some μ', then $(x_0, \mu_{\tilde{X}}(0)) \in \tilde{X}^*$; otherwise, compute $f((x_0, \mu_{\tilde{X}}(x_0)))^{\mathbf{T}} D'^{-1}$. If $((x_0, \mu_{\tilde{X}}(x_0)))^{\mathbf{T}} D'^{-1} \geqslant (\mathbf{0}, \mu')$ for some μ', then $(x_0, \mu_{\tilde{X}}(x_0)) \in \tilde{X}^*$.

Step 3. Solve the following system:

$$\begin{cases} f((s + x_0, \mu_{\tilde{X}}(s + x_0))) = (0, \mu_{\tilde{F}}(f(s + x_0))) \\ D'(s, \mu_{\tilde{X}}(s + x_0)) \geqslant (0, \mu''), D'(s, \mu_{\tilde{X}}(s + x_0)) \neq (0, \mu'') \end{cases}; \tag{7}$$

And then the solution $(s^* + x_0, \mu_{\tilde{X}}(s^* + x_0)) \in \tilde{X}^*$.

Step 4.

Step 4.1. Let $k = 1$.

Step 4.2. Let $d'_k(s, \mu_{\tilde{X}}(s+x_0)) = (0, \mu_k)$, $d'_i(s, \mu_{\tilde{X}}(s+x_0)) > (0, \mu_t)$, $\mu_k, \mu_i \in [0, 1]$ for all $i = 1, 2, \ldots, m, i \neq k$, where d_i's are row vectors of D', and $l > 0$

Step 4.3. Solve the following system to obtain the solution $(s^*, \mu_{\tilde{X}}(s^* + x_0))$.

$$\begin{aligned} & d'_k(s, \mu_{\tilde{X}}(s + x_0)) = (0, \mu_k) \\ & d'_i(s, \mu_{\tilde{X}}(s + x_0)) > (0, \mu_t) \quad i \neq k \\ & f((s + x_0, \mu_{\tilde{X}}(s + x_0)) = (l \cdot d'_k, \mu_{\tilde{F}}(f(x^*))) \end{aligned} \tag{8}$$

Then, $(s^* + x_0, \mu_{\tilde{X}}(s^* + x_0)) \in \mu^*_{\tilde{X}}$.

Step 4.4. If $k < m$, let $k = k + 1$, go back to Step 3.2; otherwise, if $n > 2$ and $m > 2$, go to Step 4; else go to Step 5.

Step 5.

Step 5.1. Let $k_1 = 1, k_2 = 2$.

Step 5.2. Solve the following system (9), and let the solution set of (9) be $\tilde{X}'$. If X' is not empty, solve the system (10) and find the solution $(s^*, \mu_{\tilde{X}}(s^* + x_0))$.

$$\begin{cases} d'_{k_1}(s, \mu_{\tilde{X}}(s + x_0)) = (0, \mu_{k_1}) \\ d'_{k_2}(s, \mu_{\tilde{X}}(s + x_0)) = (0, \mu_{k_2}) \\ d'_i(s, \mu_{\tilde{X}}(s + x_0)) > (0, \mu_i), i \neq k_1, k_2, i \in \{1, \ldots, m\} \end{cases} \tag{9}$$

$$\begin{cases} (s, \mu_{\tilde{X}}(s + x_0)) \in \mu'_{\tilde{X}} \\ f((s + x_0, \mu_{\tilde{X}}(s + x_0))) = (l_1 d'_{k_1} + l_2 d'_{k_2}, \mu_{\tilde{F}}(f(x))) \\ l_1 > 0, l_2 > 0 \end{cases} \tag{10}$$

Then $(s^* + x_0, \mu_{\tilde{X}}(s^* + x_0)) \in \tilde{X}^*$.

Step 5.3. If $k_2 \neq m$, then $k_2 = k_2 + 1$ and go to Step 4.2. If $k_2 = m$ and $k_1 < m - 1$, let $k_1 = k_1 + 1$, $k_2 = k_1 + 1$ and go to Step 4.2. If $k_2 = m$ and $k_1 = m - 1$, then go to Step 5.

Step 6. Output the optimal solution set $\tilde{X}^*$, then STOP.

In this proposed algorithm, we obtain the optimal solutions of the VI$(\tilde{X}, f)$ by solving the fuzzy systems described by (6)–(8), (10). Therefore, the main computation effort and complexity of the proposed algorithm for linear case can be referred to Table 1. When $\tilde{f}$ is nonlinear, the nonlinear system (7), (8), (10) can be rewritten as a nonlinear programming model and solved by GINO[5]. Now, we demonstrate the proposed method by some examples of VI$(\tilde{X}, f)$ below which are with respect to $\tilde{X} = \{\tilde{D}x \geqslant 0\}$ (Example 6.1), $\tilde{X} = \{Dx \underset{\sim}{\geqslant} 0\}$ (Example 6.2) and $\tilde{X} = \{\tilde{D}x \underset{\sim}{\geqslant} \tilde{0}\}$ (Example 6.3).

Table 1. The computation complexity of the proposed algorithm for linear case

Step	Condition	Computation	Complexity
Step 1	$\tilde{X} = \{\tilde{D}x \geqslant 0\}$	solve m times Max $\sum_{j=1,\, j\neq i}^{m} \left[\left(\sum_{k=1}^{n} (A_{ik} D_{jk})\right) - \left(\sum_{k=1}^{n} D_{jk}^{2}\right) \left(\sum_{k=1}^{n} A_{jk}^{2}\right) \right]$ s.t. $\mu_{\tilde{D}_{ik}}(A_{ik}) > 0, \quad k = 1, 2, \ldots, n$	$O(n^3)$ by GINO
Step 1	$\tilde{X} = \{Dx \underset{\sim}{\geqslant} 0\}$	solve $Dx = p$	$O(n^3)$
Step 1	$\tilde{X} = \{\tilde{D}x \underset{\sim}{\geqslant} 0\}$	solve m times Max $\sum_{j=1,\, j\neq i}^{m} \left[\left(\sum_{k=1}^{n} (A_{ik} D_{jk})\right) - \left(\sum_{k=1}^{n} D_{jk}^{2}\right) \left(\sum_{k=1}^{n} A_{jk}^{2}\right) \right]$ s.t. $\mu_{\tilde{D}_{ik}}(A_{ik}) > 0, \quad k = 1, 2, \ldots, n$ and solve $Dx = p$	$O(n^3)$
Step 2	D^{-1} exists, $m = n$	compute D^{-1}, $f(\tilde{x}_0)^{\mathbf{T}} D^{-1}$	$O(n^3)$
Step 2	$D \in R^{m\times n}$, $m \neq n$	solve $f(\tilde{x}_0) = D^{\mathbf{T}} \nu$	$O(mn^2), n > m+1$ $O(n^3), n = m+1$ $O\left(\binom{m+1}{n} n^3\right)$, $n < m+1$
Step 3		solve $\begin{cases} f((s+x_0, \mu_{\tilde{X}}(s+x_0))) = (0, \mu'') \\ D\tilde{s} \geqslant 0,\ D\tilde{s} \neq 0 \end{cases}$	$O((n+m)^3)$
Step 4		solve m times $\begin{cases} d_k \tilde{s} = 0 \\ d_i \tilde{s} > 0,\ i \neq k \\ \tilde{f}((s+x_0, \mu_{\tilde{X}}(s+x_0))) = (l \cdot d_k, \mu) \end{cases}$	$O(n^3 m), n > m$ $O(m^4), n < m$ $O(n^4), n = m$
Step 5		solve m times $\begin{cases} d_{k_1} \tilde{s} = 0,\ d_{k_2} \tilde{s} = 0 \\ d_i \tilde{s} > 0,\ i \neq k_1, k_2 \\ f((s+x_0, \mu_{\tilde{X}}(s+x_0))) = (l_1 d_{k_1} + l_2 d_{k_2}, \mu) \end{cases}$	$O(n^3 m), n > m$ $O(m^4), n < m$ $O(n^4), n = m$
Total	D^{-1} exists, $m = n$		$O(n^4)$
	$D \in R^{m\times n}$, $m \neq n$		$O(n^3 m), n \geqslant m+1$ $O\left(\binom{m+1}{n} n^3 + m^4\right), n < m+1$

EXAMPLE 6.1. Let $\tilde{X} = \{\tilde{x} : \tilde{5}x_1 + \tilde{12}x_2 - \tilde{7}x_3 \geqslant 0, -\tilde{4}x_1 + \tilde{3}x_2 - \tilde{5}x_3 \geqslant 0, \tilde{10}x_1 + \tilde{2}x_2 + \tilde{8}x_3 \geqslant 0, -\tilde{2}x_1 + \tilde{5}x_2 + \tilde{5}x_3 \geqslant 0, x = (x_1, x_2, x_3) \in \mathbf{R}^3\}$ and

$$f(x) = \begin{bmatrix} 2x_1 + 5x_1^3 - x_2 + 2x_3 - 4 \\ -5x_1 + x_2 + x_2^3 + 5 \\ 3x_1 - 2x_2 + 2x_3 - 1 \end{bmatrix}, \text{ solve this VI}(\tilde{X}, f).$$

$$\tilde{D} = \begin{bmatrix} \tilde{5} & \tilde{12} & -\tilde{7} \\ -\tilde{4} & \tilde{3} & -\tilde{5} \\ \tilde{10} & \tilde{2} & \tilde{8} \\ -\tilde{2} & \tilde{5} & \tilde{5} \end{bmatrix} = \begin{bmatrix} (4, 6, 1, 2) & (10, 13, 1, 1) & (-8, -5, 1, 2) \\ (-5, -2, 1, 1) & (2, 5, 2, 2) & (-4, -3, 1, 1) \\ (8, 11, 2, 1) & (1, 3, 1, 1) & (6, 9, 1, 1) \\ (-3, -1, 1, 1) & (4, 7, 1, 1) & (4, 6, 2, 1) \end{bmatrix}$$

where the fuzzy number $\tilde{D}_{ij} = (a, b, c, d)$ with the membership function

$$\mu_{\tilde{D}_{ij}}(x) = \begin{cases} 1 & \text{if } a \leqslant x \leqslant b \\ (x - a + c)/c & \text{if } a - c \leqslant x < a \\ (b + d - x)/d & \text{if } b < x \leqslant b + d \\ 0 & \text{if } x > b + d \text{ or } x < a - c \end{cases}$$

Step 0. Let $\tilde{X}^* = \emptyset$ be the optimal solution set of the VI$(\tilde{X}, f)$.

Step 1. Solve the following system and then reformulate the VI$(\tilde{X}, f)$ into GCP$'(\tilde{X}, f)$ (5), let $D' = [A_{ij}]$, $x_0 = 0$ and $b = 0$.

$$\begin{aligned} \text{Max} \quad & 4A_{11} + 8A_{12} + 8A_{13} - 212A_{11}^2 - 212A_{12}^2 - 212A_{13}^2 \\ \text{s.t.} \quad & 3 \leqslant A_{11} \leqslant 8, 9 \leqslant A_{12} \leqslant 14, -9 \leqslant A_{13} \leqslant -3 \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & 13A_{21} + 19A_{22} + 6A_{23} - 440A_{21}^2 - 440A_{22}^2 - 440A_{23}^2 \\ \text{s.t.} \quad & -6 \leqslant A_{21} \leqslant -1, 0 \leqslant A_{22} \leqslant 7, -5 \leqslant A_{23} \leqslant -2 \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & -A_{31} + 20A_{32} - 7A_{33} - 312A_{31}^2 - 312A_{32}^2 - 312A_{33}^2 \\ \text{s.t.} \quad & 6 \leqslant A_{31} \leqslant 12, 0 \leqslant A_{32} \leqslant 4, 5 \leqslant A_{33} \leqslant 10 \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & 11A_{41} + 17A_{42} - 4A_{43} - 436A_{41}^2 - 436A_{42}^2 - 436A_{43}^2 \\ \text{s.t.} \quad & -4 \leqslant A_{41} \leqslant 0, 3 \leqslant A_{42} \leqslant 8, 2 \leqslant A_{43} \leqslant 7 \end{aligned}$$

$$\text{Therefore, } D' = \begin{bmatrix} 3 & 9 & -3 \\ -1 & 0.022 & -2 \\ 6 & 0.032 & 5 \\ 0 & 3 & 2 \end{bmatrix}$$

Step 2. Solve $f((0, \mu_{\tilde{X}}(0))) = D'^{\mathbf{T}}(\nu, \mu')$,

$$\begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 6 & 0 \\ 9 & 0.022 & 0.032 & 3 \\ -3 & -2 & 5 & 2 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{bmatrix}$$

Then we can not find any $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) > 0$ that satisfies the above equation.

Step 3. Solve the following system:

$$\begin{cases} 2s_1 + 5s_1^3 - s_2 + 2s_3 - 4 = 0 \\ -5s_1 + s_2 + s_2^3 + 5 = 0 \\ 3s_1 - 2s_2 + 2s_3 - 1 = 0 \\ 3s_1 + 9s_2 - 3s_2 \geqslant 0 \\ -s_1 + 0.022s_2 - 2s_3 \geqslant 0 \\ 6s_1 + 0.032s_2 + 5s_3 \geqslant 0 \\ 3s_2 + 2s_3 \geqslant 0 \end{cases}$$

and $\mu_{\tilde{X}}(s^*) = \mathrm{Min}_i \{\mathrm{Min}_{d_i s \geqslant 0} \mathrm{Sup}_{d_i s}\{\mathrm{Min}_j \mu_{\tilde{D}}(D_{ij})\}\}$. We use GINO to solve the above nonlinear model, then no optimal solution is generated.

Step 4. For $k = 1$ to 4, solve the system (9), then we find no optimal solution.

Step 5. For $k_1 = 1$ to 3, $k_2 = k_1 - 1$ to 4 solve the systems (9) and (10), then when $k_1 = 2$, $k_2 = 4$, we can find an optimal solution $(x^*, \mu_{\tilde{X}}(x^*)) = ((0.858, 0.284, -0.426), 0.16) \in \tilde{X}^*$.

Step 6. Output the optimal solution set $\tilde{X}^* = \{(x^*, \mu_{\tilde{X}}(x^*)) : x^* = (0.858, 0.284, -0.426), \mu_{\tilde{X}}(x^*) = 0.16\}$, STOP.

EXAMPLE 6.2. Let

$$\tilde{X} = \{Dx \underset{\sim}{\geqslant} 0\} = \left\{ \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} x \underset{\sim}{\geqslant} 0 \right\}$$

$$= \left\{ (x, \mu_{\tilde{X}}(x) : \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} x \geqslant \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \right\},$$

$$f(x) = \begin{bmatrix} 2x_1 + 5x_1^3 - x_2 + 2x_3 - 4 \\ -5x_1 + x_2 + x_2^3 + 5 \\ 3x_1 - 2x_2 + 2x_3 - 1 \end{bmatrix}, \text{ solve this VI}(\tilde{X}, f).$$

Step 0. Let $\tilde{X}^* = \emptyset$ be the optimal solution set of the VI$(\tilde{X}, f)$.
Step 1. Reformulate the VI$(\tilde{X}, f)$ into GCP$'(\tilde{X}, f)$ (5), then $D' = D$, $x_0 = (7/2, -3, -3/2)^{\mathbf{T}}$ and $b = (-1, -2, -1)^{\mathbf{T}}$.
Step 2.

$$D'^{-1} = \begin{bmatrix} 2 & -3/2 & -5/2 \\ -1 & 1 & 2 \\ -1 & 1/2 & 3/2 \end{bmatrix}, \text{ then}$$

$\tilde{v}^{\mathbf{T}} = f((x_0, \mu_{\tilde{X}}(x_0)))^{\mathbf{T}} D'^{-1} = ((-12, 21/2, 37/2), \mu') < (\mathbf{0}, \mu')$. Hence $\tilde{x}_0$ is not the optimal solution.
Step 3. Solve the following system:

$$\begin{cases} 2(s_1 + 7/2) + 5(s_1 + 7/2)^3 - (s_2 - 3) + 2(s_3 - 3/2) - 4 = 0 \\ -5(s_1 + 7/2) + (s_2 - 3) + (s_2 - 3)^3 + 5 = 0 \\ 3(s_1 + 7/2) - 2(s_2 - 3) + 2(s_3 - 3/2) - 1 = 0 \\ s_1 + 2s_2 - s_3 \geqslant 0 \\ -s_1 + s_2 - 3s_3 \geqslant 0 \\ s_1 + s_2 + s_3 \geqslant 0 \\ x = (x_1, x_2, x_3) = (s_1 + 7/2, s_2 - 3, s_3 - 3/2) \end{cases}$$

and $\mu_{\tilde{X}}(x) = \text{Min}_i\{\mu_i(x)\}$ where

$$\mu_i(x) = \begin{cases} 0 & \text{if } d_i'x \leqslant b_i \\ 1 - d_i'x/b_i & \text{if } 0 > d_i'x > b_i \\ 1 & \text{if } d_i'x \geqslant 0 \end{cases}.$$

We use GINO to solve the above nonlinear model and obtain the optimal solution $(x^*, \mu_{\tilde{X}}(x^*)) = ((0.994, -0.262, -1.178), 0.554) \in \tilde{X}^*$.
Step 4. For $k = 1$ to 3, solve the system (8), then we find no optimal solution.
Step 5. For $k_1 = 1$ to 2, $k_2 = k_1 - 1$ to 3 solve the systems (9) and (10), then we find no optimal solution.
Step 6. Output the optimal solution set $\tilde{X}^* = \{(x^*, \mu_{\tilde{X}}(x^*)) : x^* = (0.994, -0.262, -1.178), \mu_{\tilde{X}}(x^*) = 0.554\}$, STOP.

EXAMPLE 6.3. Let $\tilde{X} = \{\tilde{x} : \tilde{5}x_1 + \tilde{12}x_2 - \tilde{7}x_3 \underset{\sim}{\geqslant} \tilde{0}, -\tilde{4}x_1 + \tilde{3}x_2 - \tilde{5}x_3 \underset{\sim}{\geqslant} \tilde{0}, \tilde{10}x_1 + \tilde{2}x_2 + \tilde{8}x_3 \underset{\sim}{\geqslant} \tilde{0}, -\tilde{2}x_1 + \tilde{5}x_2 + \tilde{5}x_3 \underset{\sim}{\geqslant} \tilde{0}, x = (x_1, x_2, x_3) \in \mathbf{R}^3\} =$

$$\left\{ \begin{bmatrix} \tilde{5} & \tilde{12} & -\tilde{7} \\ -\tilde{4} & \tilde{3} & -\tilde{5} \\ \tilde{10} & \tilde{2} & \tilde{8} \\ -\tilde{2} & \tilde{5} & \tilde{5} \end{bmatrix} x \geqslant \begin{bmatrix} -5 \\ -5 \\ -2 \\ -1 \end{bmatrix} \right\},$$

$$\tilde{D} = \begin{bmatrix} \tilde{5} & \tilde{12} & -\tilde{7} \\ -\tilde{4} & \tilde{3} & -\tilde{5} \\ \tilde{10} & \tilde{2} & \tilde{8} \\ -\tilde{2} & \tilde{5} & \tilde{5} \end{bmatrix} = \begin{bmatrix} (4, 6, 1, 2) & (10, 13, 1, 1) & (-8, -5, 1, 2) \\ (-5, -2, 1, 1) & (2, 5, 2, 2) & (-4, -3, 1, 1) \\ (8, 11, 2, 1) & (1, 3, 1, 1) & (6, 9, 1, 1) \\ (-3, -1, 1, 1) & (4, 7, 1, 1) & (4, 6, 2, 1) \end{bmatrix}$$

and

$$f(x) = \begin{bmatrix} 2x_1 + 5x_1^3 - x_2 + 2x_3 - 4 \\ -5x_1 + x_2 + x_2^3 + 5 \\ 3x_1 - 2x_2 + 2x_3 - 1 \end{bmatrix}, \text{ solve this VI}(\tilde{X}, f)$$

Step 0. Let $\tilde{X}^* = \emptyset$ be the optimal solution set of the VI$(\tilde{X}, f)$.

Step 1. Solve the following system and then reformulate the VI$(\tilde{X}, f)$ into GCP$'(\tilde{X}, f)$ (5), let $D' = [A_{ij}]$,

$$b = \begin{bmatrix} -5 \\ -5 \\ -2 \\ -1 \end{bmatrix}$$

and solve $D'x_0 = b$.

$$\begin{aligned} \text{Max} \quad & 4A_{11} + 8A_{12} + 8A_{13} - 212A_{11}^2 - 212A_{12}^2 - 212A_{13}^2 \\ \text{s.t.} \quad & 3 \leqslant A_{11} \leqslant 8, 9 \leqslant A_{12} \leqslant 14, -9 \leqslant A_{13} \leqslant -3 \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & 13A_{21} + 19A_{22} + 6A_{23} - 440A_{21}^2 - 440A_{22}^2 - 440A_{23}^2 \\ \text{s.t.} \quad & -6 \leqslant A_{21} \leqslant -1, 0 \leqslant A_{22} \leqslant 7, -5 \leqslant A_{23} \leqslant -2 \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & -A_{31} + 20A_{32} - 7A_{33} - 312A_{31}^2 - 312A_{32}^2 - 312A_{33}^2 \\ \text{s.t.} \quad & 6 \leqslant A_{31} \leqslant 12, 0 \leqslant A_{32} \leqslant 4, 5 \leqslant A_{33} \leqslant 10 \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & 11A_{41} + 17A_{42} - 4A_{43} - 436A_{41}^2 - 436A_{42}^2 - 436A_{43}^2 \\ \text{s.t.} \quad & -4 \leqslant A_{41} \leqslant 0, 3 \leqslant A_{42} \leqslant 8, 2 \leqslant A_{43} \leqslant 7 \end{aligned}$$

$$\text{Therefore,} \; D' = \begin{bmatrix} 3 & 9 & -3 \\ -1 & 0.022 & -2 \\ 6 & 0.032 & 5 \\ 0 & 3 & 2 \end{bmatrix} \text{ and } x_0 = (0.745, -0.221, 1.749).$$

Step 2. Solve $f((x_0, \mu_{\tilde{X}}(x_0))) = D^{'\mathbf{T}}(\nu, \mu')$,

$$\begin{bmatrix} 3.276 \\ 1.043 \\ 5.175 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 6 & 0 \\ 9 & 0.022 & 0.032 & 3 \\ -3 & -2 & 5 & 2 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{bmatrix}$$

Then we can not find any $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) > 0$ that satisfies the above equation.

Step 3. Solve the following system:

$$\begin{cases} 2(s_1 + 0.745) + 5(s_1^3 + 0.745) - (s_2 - 0.221) + 2(s_3 + 1.749) - 4 = 0 \\ -5(s_1 + 0.745) + (s_2 - 0.221) + (s_2 + 10749)^3 + 5 = 0 \\ 3(s_1 + 0.745) - 2(s_2 - 0.221) + 2(s_3 + 1.749) - 1 = 0 \\ 3s_1 + 9s_2 - 3s_3 \geqslant 0 \\ -s_1 + 0.022s_2 - 2s_3 \geqslant 0 \\ 6s_1 + 0.032s_2 + 5s_3 \geqslant 0 \\ 3s_2 + 2s_3 \geqslant 0 \end{cases}$$

and $\mu_{\tilde{X}}(s^*) = \text{Min}_i \{\text{Min}_{d_i s \geqslant 0} \text{Sup}_{d_i s}\{\text{Min}_j \mu_{\tilde{D}}(D_{ij})\}\}$. We use GINO to solve the above nonlinear model, then no optimal solution is generated.

Step 4. For $k = 1$ to 4, solve the system (8), then we find no optimal solution.

Step 5. For $k_1 = 1$ to 3, $k_2 = k_1 - 1$ to 4 solve the systems (9) and (10), then when $k_1 = 2$, $k_2 = 4$, we can find an optimal solution $(x^*, \mu_{\tilde{X}}(x^*)) = ((0.204, 0.164, -0.246), 0.16) \in \tilde{X}^*$.

Step 6. Output the optimal solution set $\tilde{X}^* = \{(x^*, \mu_{\tilde{X}}(x^*)) : x^* = (0.204, 0.164, -0.246), \mu_{\tilde{X}}(x^*) = 0.33\}$, STOP.

7. Summary and conclusion

In this study, we proposed an approach to solving variational inequality problems with fuzzy convex cone from the concepts of multiple objective programming and fuzzy set theory. With different classes of fuzzy convex cones, the fuzzy optimal solution sets were derived. An algorithm was developed to find the optimal fuzzy solution set for general case, and the computational complexity of linear case is polynomial. Theoretical evidences were illustrated by numerical examples.

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